USM-TH-165

On external backgrounds and linear potential in three dimensions

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(Dated: February 1, 2008)

For a three-dimensional theory with a coupling $\phi\varepsilon^{\mu\nu\lambda}v_{\mu}F_{\nu\lambda}$, where v_{μ} is an external constant background, we compute the interaction potential within the structure of the gauge-invariant but path-dependent variables formalism. While in the case of a purely timelike vector the static potential remains Coulombic, in the case of a purely spacelike vector the potential energy is the sum of a Bessel and a linear potentials, leading to the confinement of static charges. This result may be considered as another realization of the known Polyakov's result.

PACS numbers: 11.10.Ef, 11.15.-q

I. INTRODUCTION

It is presently widely accepted that quantitative understanding of confinement remains as the major challenge in QCD. In this respect, the distinction between the apparently related phenomena of screening and confinement have been of considerable importance in order to gain further insight and theoretical guidance into this problem. It is worth recalling at this stage that field theories which yield a linear potential are relevant to particle physics, since those theories may be used to understand the confinement of quarks and be considered as effective theories of QCD. According to this viewpoint, a simple effective theory in which confining potentials are obtained when a scalar field ϕ is coupled to gauge fields via the $3+1$ dimensional interaction term

$$\mathcal{L}_I = \frac{g}{8}\phi\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}, \tag{1}$$

has been investigated in Ref.[1]. The phenomenology of this model has shown that in the case of a constant electric field strength expectation value the static potential remains Coulombic, while in the case of a constant magnetic field strength the potential energy is the sum of a Yukawa and a linear potential, leading to the confinement of static charges. More interestingly, similar results have been obtained in the context of the dual Ginzburg-Landau theory [2], as well as for a theory of antisymmetric tensor fields that results from the condensation of topological defects as a consequence of the Julia-Toulouse mechanism [3]. Thus, from a phenomenological point of view, there is a class of models which are good candidates for effective theories of QCD.

In order to give continuity to this program it would be interesting to verify the above scenario when a scalar field ϕ is coupled to gauge fields via the $2+1$ dimensional interaction term

$$\mathcal{L}_I = \frac{g}{2}\phi\varepsilon^{\mu\nu\lambda}v_{\mu}F_{\nu\lambda}, \tag{2}$$

where v_{μ} is an external constant background. Our motivation is mainly to explore the effects of the external background constant on the confining and screening nature of the potential that could be useful for later developments in more realistic theories, such as describing low-energy properties in condensed matter physics [4]. Here we should mention that the interaction term (2) arises from the dimensional reduction of Maxwell electrodynamics with the (*Lorentz*-violating) *Carroll-Field-Jackiw* term [5, 6]. The *Lorentz* violating theories have been extensively discussed in the last few years. For example, in connection with ultra-high energy cosmic rays [7], with space-time varying coupling constants [8], and in supersymmetric *Lorentz*-violating extensions [9]. In this Letter, however, we examine the effects of the *Lorentz* violating background on the interaction energy along the lines of Refs.[1, 3]. It must now be observed that this approach provides a physically-based alternative to the usual Wilson loop approach, where in the latter the usual qualitative picture of confinement in terms of an electric flux tube linking quarks emerges naturally. As we shall see, the static potential remains Coulombic for a purely timelike vector v^{μ}. On the other hand, adopting a purely spacelike vector v^{μ}, the potential energy is the sum of a Bessel and a linear potential, that is, the confinement between static charges is obtained. In this last respect we recall that, almost twenty years ago, *Polyakov* [10] showed that compact Maxwell theory in $2+1$ dimensions confines permanently electric test charges. In this way

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our calculation provides a complementary phenomenological picture of the known Polyakov's result, in the hope that this will be helpful to understand better effective gauge theories in $2+1$ dimensions.

II. INTERACTION ENERGY

As mentioned above, our principal purpose is to calculate explicitly the interaction energy between static pointlike sources for a model one which contains the term (2). This model is similar to the studied in Ref.[1]. To this end we will compute the expectation value of the energy operator H in the physical state $|\Phi\rangle$ describing the sources, which we will denote by $\langle H\rangle_\Phi$. The Abelian gauge theory we are considering is defined by the following generating functional in three-dimensional spacetime:

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}A \exp\left\{ i\int d^3x\mathcal{L}\right\}, \tag{3}$$

where the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{g}{2}\phi\varepsilon^{\mu\nu\lambda}v_\mu F_{\nu\lambda} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2, \tag{4}$$

and m is the mass for the scalar field ϕ. As in [1] we restrict ourselves to static scalar fields, a consequence of this is that one may replace $\Delta\phi = -\nabla^2\phi$, with $\Delta \equiv \partial_\mu\partial^\mu$. It also implies that, after performing the integration over ϕ in $\mathcal{Z}$, the effective Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{g^2}{8}\varepsilon^{\mu\nu\lambda}v_\mu F_{\nu\lambda}\frac{1}{\nabla^2 - m^2}\varepsilon^{\sigma\gamma\beta}v_\sigma F_{\gamma\beta}. \tag{5}$$

By introducing $V^{\nu\lambda} \equiv \varepsilon^{\mu\nu\lambda}v_\mu$, expression (5) then becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{g^2}{8}V^{\nu\lambda}F_{\nu\lambda}\frac{1}{\nabla^2 - m^2}V^{\gamma\beta}F_{\gamma\beta}. \tag{6}$$

At this stage we note that (6) has the same form as the corresponding effective Lagrangian density in four-dimensional spacetime. This common feature provides the starting point for the examination of the effects of the *Lorentz* violating background on the interaction energy.

A. Spacelike background case

We now proceed to obtain the interaction energy in the $V^{0i} \neq 0$ and $V^{ij} = 0$ ($v_0 = 0$) case (referred to as the spacelike background in what follows), by computing the expectation value of the Hamiltonian in the physical state $|\Phi\rangle$. Lagrangian density (6) then becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{g^2}{2}V^{0i}F_{0i}\frac{1}{\nabla^2 - m^2}V^{0k}F_{0k} - A_0 J^0, \tag{7}$$

where J^0 is the external current, $(\mu,\nu = 0,1,2)$ and $(i,k = 1,2)$.

Once this is done, the canonical quantization in the manner of *Dirac* yields the following results. The canonical momenta are

$$\Pi^0 = 0, \tag{8}$$

and

$$\Pi_i = D_{ij}E_j, \tag{9}$$

where $E_i \equiv F_{i0}$ and $D_{ij} \equiv \left(\delta_{ij} - g^2 V_{i0}\frac{1}{\nabla^2 - m^2}V_{j0}\right)$. Since D is a nonsingular matrix ($\det D = 1 - g^2\frac{\mathbf{V}^2}{\nabla^2 - m^2} \neq 0$) with $\mathbf{V}^2 \equiv V^{i0}V^{i0}$, there exists the inverse of D and from Eq.(9) we obtain

$$E_i = \frac{1}{\det D}\left\{\delta_{ij}\det D + g^2 V_{i0}\frac{1}{\nabla^2 - m^2}V_{j0}\right\}\Pi_j. \tag{10}$$

The corresponding canonical Hamiltonian is thus

$$H_C = \int d^2x \left\{ -A_0 \left(\partial_i \Pi^i - J^0\right) + \frac{1}{2}\mathbf{\Pi}^2 + \frac{g^2}{2}\frac{(\mathbf{V}\cdot\mathbf{\Pi})^2}{(\nabla^2 - M^2)} + \frac{1}{2}B^2 \right\}, \tag{11}$$

where $M^2 \equiv m^2 + g^2 V^2$ and B is the magnetic field. Requiring the primary constraint (8) to be stationary, leads to the secondary constraint $\Gamma_1(x) \equiv \partial_i \Pi^i - J^0 = 0$. It is easily verified that the preservation of Γ_1 for all times does not give rise to any further constraints. The theory is thus seen to possess only two constraints, which are first class. The extended Hamiltonian that generates translations in time then reads $H = H_C + \int d^2x \left(c_0(x)\Pi_0(x) + c_1(x)\Gamma_1(x)\right)$, where $c_0(x)$ and $c_1(x)$ are the Lagrange multiplier fields. Since $\Pi_0 = 0$ for all time and $\dot{A}_0(x) = [A_0(x), H] = c_0(x)$, which is completely arbitrary, we discard $A_0(x)$ and $\Pi_0(x)$ because they add nothing to the description of the system. Then, the Hamiltonian takes the form

$$H = \int d^2x \left\{ \frac{1}{2}\mathbf{\Pi}^2 + \frac{g^2}{2}\frac{(\mathbf{V}\cdot\mathbf{\Pi})^2}{(\nabla^2 - M^2)} + \frac{1}{2}B^2 + c(x)\left(\partial_i \Pi^i - J^0\right) \right\}, \tag{12}$$

where $c(x) = c_1(x) - A_0(x)$.

The quantization of the theory requires the removal of non-physical variables, which is done by imposing a gauge condition such that the full set of constraints becomes second class. A convenient choice is found to be [11]

$$\Gamma_2(x) \equiv \int\limits_{C_{\xi x}} dz^\nu A_\nu(z) \equiv \int\limits_0^1 d\lambda x^i A_i(\lambda x) = 0, \tag{13}$$

where λ $(0 \leq \lambda \leq 1)$ is the parameter describing the spacelike straight path $x^i = \xi^i + \lambda (x-\xi)^i$, and ξ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. The Dirac brackets can now be obtained and the nontrivial Dirac bracket involving the field variables takes the form

$$\left\{A_i(x), \Pi^j(y)\right\}^* = \delta_i^j \delta^{(2)}(x-y) - \partial_i^x \int\limits_0^1 d\lambda x^j \delta^{(2)}(\lambda x - y). \tag{14}$$

We are now in a position to evaluate the interaction energy between pointlike sources in the model under consideration, where a fermion is localized at $\mathbf{y\prime}$ and an antifermion at $\mathbf{y}$. From our above discussion, we see that $\langle H\rangle_\Phi$ reads

$$\langle H\rangle_\Phi = \langle\Phi| \int d^2x \left\{ \frac{1}{2}\mathbf{\Pi}^2 + \frac{g^2}{2}\frac{(\mathbf{V}\cdot\mathbf{\Pi})^2}{(\nabla^2 - M^2)} + \frac{1}{2}B^2 \right\} |\Phi\rangle. \tag{15}$$

Next, as was first established by Dirac[12], the physical state can be written as

$$|\Phi\rangle \equiv \left|\overline{\Psi}(\mathbf{y})\Psi(\mathbf{y\prime})\right\rangle = \overline{\psi}(\mathbf{y}) \exp\left(ie \int\limits_{\mathbf{y\prime}}^{\mathbf{y}} dz^i A_i(z)\right) \psi(\mathbf{y\prime})|0\rangle, \tag{16}$$

where $|0\rangle$ is the physical vacuum state and the line integral appearing in the above expression is along a spacelike path starting at $\mathbf{y\prime}$ and ending at $\mathbf{y}$, on a fixed time slice. From this we see that the fermion fields are now dressed by a cloud of gauge fields. From the foregoing Hamiltonian discussion, we first note that

$$\Pi_i(x)\left|\overline{\Psi}(\mathbf{y})\Psi(\mathbf{y}')\right\rangle = \overline{\Psi}(\mathbf{y})\Psi(\mathbf{y}')\Pi_i(x)|0\rangle + e\int_{\mathbf{y}}^{\mathbf{y}'} dz_i \delta^{(3)}(\mathbf{z}-\mathbf{x})|\Phi\rangle. \tag{17}$$

Combining Eqs.(15) and (17), we have

$$\langle H\rangle_\Phi = \langle H\rangle_0 + V^{(1)} + V^{(2)}, \tag{18}$$

where $\langle H\rangle_0 = \langle 0|H|0\rangle$, and the $V^{(1)}$ and $V^{(2)}$ terms are given by:

$$V^{(1)} = -\frac{e^2}{2}\int d^2x \int_{\mathbf{y}}^{\mathbf{y}'} dz_i' \delta^{(2)}(x-z') \frac{1}{\nabla_x^2 - M^2}\nabla_x^2 \int_{\mathbf{y}}^{\mathbf{y}'} dz^i \delta^{(2)}(x-z), \tag{19}$$

and

$$V^{(2)} = \frac{e^2 m^2}{2} \int d^2x \int_{\mathbf{y}}^{\mathbf{y}'} dz_i' \delta^{(2)}(x - z') \frac{1}{\nabla_x^2 - M^2} \int_{\mathbf{y}}^{\mathbf{y}'} dz^i \delta^{(2)}(x - z), \tag{20}$$

where the integrals over z^i and z_i' are zero except on the contour of integration.

The $V^{(1)}$ term may look peculiar, but it is just the familiar Bessel interaction plus self-energy terms. In effect, expression (19) can also be written as

$$V^{(1)} = \frac{e^2}{2} \int_{\mathbf{y}}^{\mathbf{y}'} dz_i' \partial_i^{z'} \int_{\mathbf{y}}^{\mathbf{y}'} dz^i \partial_z^i G(\mathbf{z}', \mathbf{z}), \tag{21}$$

where G is the Green function

$$G(\mathbf{z}', \mathbf{z}) = \frac{1}{2\pi} K_0(M|\mathbf{z}' - \mathbf{z}|). \tag{22}$$

Employing Eq. (22) and remembering that the integrals over z^i and z_i' are zero except on the contour of integration, expression (21) reduces to the familiar Bessel interaction after subtracting the self-energy terms, that is,

$$V^{(1)} = -\frac{e^2}{2\pi} K_0(M|\mathbf{y} - \mathbf{y}'|). \tag{23}$$

The task is now to evaluate the $V^{(2)}$ term, which is given by

$$V^{(2)} = \frac{e^2 m^2}{2} \int_{\mathbf{y}}^{\mathbf{y}'} dz'^i \int_{\mathbf{y}}^{\mathbf{y}'} dz^i G(\mathbf{z}', \mathbf{z}). \tag{24}$$

By using the integral representation of the Green function (22)

$$K_0(x) = \int_0^\infty \cos(x \sinh t) dt = \int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}} dt, \tag{25}$$

where $x > 0$, expression (24) can also be written as

$$V^{(2)} = \frac{e^2 m^2}{2\pi M^2} \int_0^\infty dt \frac{1}{t^2} \frac{1}{\sqrt{t^2+1}} (1 - \cos(MLt)), \tag{26}$$

where $L \equiv |\mathbf{y} - \mathbf{y}'|$.

Now let us calculate integral (26). For this purpose we introduce a new auxiliary parameter ε by making in the denominator of integral (26) the substitution $t^2 \to t^2 + \varepsilon^2$. Thus it follows that

$$V^{(2)} \equiv \lim_{\varepsilon \to 0} \widetilde{V}^{(2)} = \lim_{\varepsilon \to 0} \frac{e^2 m^2}{2\pi M^2} \int_0^\infty \frac{dt}{t^2 + \varepsilon^2} \frac{1}{\sqrt{t^2+1}} (1 - \cos(MLt)). \tag{27}$$

A direct computation on the t-complex plane yields

$$\widetilde{V}^{(2)} = \frac{e^2 m^2}{4M^2} \left(\frac{1 - \mathrm{e}^{-ML\varepsilon}}{\varepsilon} \right) \frac{1}{\sqrt{1 - \varepsilon^2}}. \tag{28}$$

Taking the limit $\varepsilon \to 0$, expression (28) then becomes

$$V^{(2)} = \frac{e^2 m^2}{4M} |\mathbf{y} - \mathbf{y}'|. \tag{29}$$

From Eqs.(23) and (29), the corresponding static potential for two opposite charges located at $\mathbf{y}$ and $\mathbf{y}'$ may be written as

$$V(L) = -\frac{e^2}{2\pi} K_0(ML) + \frac{e^2 m^2}{4M} L, \tag{30}$$

where $L \equiv |\mathbf{y} - \mathbf{y}'|$.

It must now be observed that the rotational symmetry is restored in the resulting form of the potential, although the external background breaks the isotropy of the problem in a manifest way. It should be remarked that this feature is also shared by the corresponding four-dimensional spacetime interaction energy.

We further note that the result (30) agrees with that of Polyakov based on the monopole plasma mechanism, in the short distance regime. In this way the above analysis reveals that, although both models are different, the physical content is identical in the short distance regime. This behavior is also obtained in the context of the condensation of topological defects [13].

B. Timelike background case

Now we focus on the case $V^{0i} = 0$ and $V^{ij} \neq 0$ ($v_0 \neq 0$) (referred to as the timelike background in what follows). The corresponding Lagrangian density reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{g^2}{8} V^{ij} F_{ij} \frac{1}{\nabla^2 - m^2} V^{kl} F_{kl} - A_0 J^0, \tag{31}$$

$(\mu, \nu = 0, 1, 2)$ and $(i, j, k, l = 1, 2)$.

Here again, the quantization is carried out using Dirac's procedure. We can thus write the canonical momenta $\Pi^\mu = F^{\mu 0}$, which results in the usual primary constraint $\Pi^0 = 0$ and $\Pi^i = F^{i0}$. Defining the electric and magnetic fields, as usual, by $E^i = F^{i0}$ and $B^2 = \frac{1}{2} F_{ij} F^{ij}$, the canonical Hamiltonian is thus

$$H_C = \int d^2x \left\{ \frac{1}{2}\mathbf{E}^2 + \frac{1}{2} B^2 - \frac{g^2}{16} \varepsilon_{ijm} \varepsilon_{kln} V^{ij} B^m \frac{1}{\nabla^2 - m^2} V^{kl} B^n - A_0 \left(\partial_i \Pi^i - J^0 \right) \right\}. \tag{32}$$

Time conservation of the primary constraint $\Pi^0 = 0$ leads to the secondary constraint $\Gamma_1(x) \equiv \partial_i \Pi^i - J^0 = 0$, and the preservation of $\Gamma_1(x)$ for all times does not give rise to any further constraints. Moreover, it is straightforward to see that the constrained structure for the gauge field is identical to the usual Maxwell theory. However, in order to put the discussion into the context of this paper, it is convenient to summarize the relevant aspects of the analysis described previously[14]. Therefore, we pass now to the calculation of the interaction energy.

As in the previous subsection, our objective will be to calculate the expectation value of the Hamiltonian in the physical state $|\Phi\rangle$ (Eq. (16)). In other words,

$$\langle H \rangle_\Phi = \langle \Phi | \int d^2x \left\{ \frac{1}{2} \mathbf{E}^2 \right\} |\Phi\rangle . \tag{33}$$

Then using the above Hamiltonian structure, we obtain the following form:

$$\langle H \rangle_\Phi = \langle H \rangle_0 + \frac{e^2}{2} \int d^2x \left(\int_{\mathbf{y}}^{\mathbf{y}'} dz_i \delta^{(2)} \left(x - z \right) \right)^2, \tag{34}$$

where $\langle H \rangle_0 = \langle 0 \mid H \mid 0 \rangle$ and, as before, the integrals over z_i are zero except on the contour of integrations. We also draw attention to the fact that the second term on the right-hand side of Eq.(34) is clearly dependent on the distance between the external static fields. In fact, this term can be manipulated in a similar manner to that in the three-dimensional case [15]. Accordingly, the potential for two opposite charges located at $\mathbf{y}$ and $\mathbf{y}\prime$ reads

$$V = \frac{e^2}{2\pi} \ln(\mu L), \tag{35}$$

where μ is a massive cutoff introduced to regularize the potential, and $L \equiv |\mathbf{y} - \mathbf{y}'|$.

III. FINAL REMARKS

In summary, we have considered the confinement versus screening issue for a three-dimensional theory with a coupling $\phi \varepsilon^{\mu\nu\lambda} v_\mu F_{\nu\lambda}$, when the external constant vector v_μ is pure spacelike or timelike, respectively.

It was shown that in the case when the vector v_μ is purely timelike no unexpected features are found. Indeed, the resulting static potential remains Coulombic. More interestingly, it was shown that when the vector v_μ is purely spacelike the static potential displays a Bessel piece plus a linear confining piece. Effectively, therefore, the model studied leads to a confining potential between static charges for spacelike v_μ. An analogous situation in the four-dimensional spacetime case may be recalled [1], where a constant expectation value for the gauge field strength $\langle F_{\mu\nu}\rangle$ characterizes the external background constant v_μ. Also, a common feature of these models (three and four-dimensional) is that the rotational symmetry is restored in the resulting interaction energy.

We conclude by noting that our result can be considered as another physical realization of the Polyakov's model. However, although both models lead to confinement, the above analysis reveals that the mechanism of obtaining a linear potential is quite different.

IV. ACKNOWLEDGMENTS

I would like to thank G. Cvetic for helpful comments on the manuscript. Work supported in part by Fondecyt (Chile) grant 1050546.

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